# Growth of Sobolev norms for the NLS on $\mathbb{T}^2$

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# The problem

## The equation

- We consider the defocusing Schrödinger equation on the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$

$$\begin{cases} -i\partial_t u + \Delta u = |u|^{2p} u, & x \in \mathbb{T}^2, \\ u(0,x) = u_0(x) \end{cases} \qquad (NLS)$$

- a very reasonable question is to study the time evolution of the Sobolev norms

$$\int_{\mathbb{T}^2} |(1-\Delta)^{s/2} u|^2$$

- growth of Sobolev norms ($s > 1$): forward cascade of energy

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$p = 1$ is the cubic NLS, $p = 2$ the quintic...
you can consider any analytic non-linearity

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- COMPACT MANIFOLD: one does not expect scattering
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- Two conserved quantities are the Hamiltonian

$$H = \int_{\mathbb{T}^2} |\nabla u|^2 + \frac{1}{p+1}\int_{\mathbb{T}^2} |u|^{2p+2}$$

and the mass (the $L^2$ norm)

$$L = \int_{\mathbb{T}^2} |u|^2$$

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$$\int |(1-\Delta)^{s/2} \ldots|^2$$

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# Basic properties: $-i\partial_t u + \Delta u = |u|^{2p} u$

If we pass to the Fourier modes:

$$u(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{in\cdot x}$$

the NLS can be seen as an infinite chain of harmonic oscillators coupled by the non-linearity.

$$H = \sum_{n \in \mathbb{Z}^2} |n|^2 |a_n|^2 + \frac{1}{p+1} \sum_{\substack{n_1,\dots,n_{2p+2} \in \mathbb{Z}^2 \\ \sum (-1)^i n_i = 0}} a_{n_1} \bar{a}_{n_2} \dots a_{n_{2p+1}} \bar{a}_{n_{2p+2}}$$

The linear NLS is of completely resonant (namely all the harmonic oscillators have the same linear frequency)

# Basic properties: $-i\partial_t u + \Delta u = |u|^{2p} u$

the linear flow:

- preserves the actions $|a_n|^2$
- all the linear solutions are periodic

When we take into account the non-linearity the solutions should be much more complicated, We EXPECT:

- Existence of periodic, quasi-periodic and almost-periodic solutions.
- Existence of solutions for which the linear actions $|a_n|^2$ are not approximately preserved.
- Solutions which are initially supported only on low Fourier modes and eventually transfer energy to arbitrarily high modes (weak turbulence).
- Growth of Sobolev (semi-)norms:

$$\sum_{j \in \mathbb{Z}^2} |j|^{2s} |a_j|^2$$

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# Basic properties: $-i\partial_t u + \Delta u = |u|^{2p} u$

- global well-posedness in $H^s(\mathbb{T}^2)$, $s \geq 1$
- smooth solution for all times from smooth initial data.
- For (defocusing) polynomial nonlinearities of the type $\partial_{\bar{u}} P(|u|^2)$, the Sobolev norms ($s > 1$) grow at most polynomially in time (Bourgain '96, Staffilani '97 and later improvements)

# Growth of Sobolev norms in NLS on $\mathbb{T}^d$

- Several results giving upper bounds on the growth, few results obtaining existence of growth
- Breakthrough results by
  Colliander-Keel-Staffilani-Takaoka-Tao 2010: growth (of a finite but arbitrarily large factor) of Sobolev norms for the two-dimensional cubic equation
  Kaloshin-Guardia 2013: growth of Sobolev norms with control on the time.

# Literature on related equations

- Gérard-Grellier on the half-wave equation and the cubic Szegő equation
- Hani-Pausader-Tzvetkov-Visciglia unbounded Sobolev orbits for the cubic NLS on $\mathbb{T}^2 \times \mathbb{R}$

# Our main result: $-i\partial_t u + \Delta u = |u|^{2p} u$

**Theorem (Guardia-H-Procesi)**

*$p \geq 1$ and $s > 1$. Then, for any large $K \gg 1$ and $\delta \ll 1$, there exists a global solution $u(t) = u(t, \cdot)$ of (NLS) such that*

$$\|u(0)\|_{H^s} \leq \delta \quad \text{and} \quad \|u(T)\|_{H^s} \geq K.$$

For $p = 1$ this is the result [CKSTT]!

We can also give estimates on the time

$$T \leq \left(\frac{K}{\delta}\right)^{16p\frac{K}{\delta}\ln\left(\frac{K}{\delta}\right)}$$

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# The 1-dimensional case

it would be interesting to study

$$-i\partial_t u + u_{xx} = |u|^{2p} u$$

on a circle, for $p > 1$
(for $p = 1$, the 1d cubic NLS is completely integrable).

at present we have only negative results

# Structure of the proof

- Combinatorial part: construction of a finite dimensional "toy model"
  construct a finite dimensional Hamiltonian system which approximates the NLS.
- Dynamical part: proof of the existence of diffusing orbits in the toy model
- Analytical part: approximation lemma and time estimates

while the analytical part is an adaptation of the ideas and techniques used for the cubic case in [CKSTT] and [Guardia-Kaloshin], the combinatorial part is new

## The Birkhoff Normal Form

$$u(t,x) = \sum_{n\in\mathbb{Z}^2} a_n(t) e^{in\cdot x}$$

- The Hamiltonian in the Fourier coefficients $a_n$:

$$H = \sum_{n\in\mathbb{Z}^2} |n|^2 |a_n|^2 + \frac{1}{p+1} \sum_{\substack{n_1,\dots,n_{2p+2}\in\mathbb{Z}^2 \\ \sum(-1)^i n_i = 0}} a_{n_1}\bar{a}_{n_2}\dots a_{n_{2p+1}}\bar{a}_{n_{2p+2}}$$

- First step of Birkhoff normal form: degree $2p+2$ nonresonant terms are removed from the Hamiltonian

$$H = H^{(2)} + H^{(2p+2)} \longrightarrow H^{(2)} + H^{(2p+2)}_{Res} + H^{(4p+2)}$$

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- Resonant terms are terms of degree $2p+2$ which Poisson commute with the quadratic term

$$\sum_{n \in \mathbb{Z}^2} |n|^2 |a_n|^2$$

## The Birkhoff Normal Form

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- By taking $u$ small enough we can ignore the term $H^{(4p+2)}$ (for finite but long time!) and study the dynamics of

$$H_{Birk} = H^{(2)} + H^{(2p+2)}_{Res}$$

# The Birkhoff Normal Form

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- actually since $H^{(2)}$ is a constant of motion we just study

$$H_{Birk} = H^{(2p+2)}_{Res}$$

# Resonances

$$H_{Birk} = \frac{1}{p+1} \sum_{\substack{n_1,\dots,n_{2p+2}\in\mathbb{Z}^2 \\ \sum(-1)^i n_i=0 \\ \sum(-1)^i |n_i|^2=0}} a_{n_1}\bar{a}_{n_2}\dots a_{n_{2p+1}}\bar{a}_{n_{2p+2}}$$

The sum is restricted to resonances:

$$n_1 - n_2 + n_3 + \cdots + n_{2p+1} - n_{2p+2} = 0\,,$$

$$|n_1|^2 - |n_2|^2 + |n_3|^2 + \cdots + |n_{2p+1}|^2 - |n_{2p+2}|^2 = 0$$

# Resonances: the cubic case

in the case of the cubic NLS resonances have a geometric interpretation:

$$n_1 - n_2 + n_3 - n_4 = 0\,, \qquad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0$$

are rectangles :

$$\begin{array}{cc} \circ\, \mathtt{n}_1 & \circ\, \mathtt{n}_4 \\ & \\ \circ\, \mathtt{n}_2 & \circ\, \mathtt{n}_3 \end{array}$$

# Invariant subspaces

$$H_{Birk} = \frac{1}{p+1} \sum_{\substack{n_1,\dots,n_{2p+2}\in\mathbb{Z}^2 \\ \sum(-1)^i n_i=0 \\ \sum(-1)^i |n_i|^2=0}} a_{n_1}\bar{a}_{n_2}\dots a_{n_{2p+1}}\bar{a}_{n_{2p+2}}$$

Has a complicated dynamics but many invariant subspaces...

We consider a set $\Lambda = \{j_1,\dots,j_m\}$, $j_i \in \mathbb{Z}^2$ and look for invariant subspaces of the form:

$$U_\Lambda := \{a_j = 0\,, \quad \forall j \notin \Lambda\}$$

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I want to find a finite dimensional subspace on which there is "growth of Sobolev norms"

# Strategy of the proof

- Choice of the initial datum within a finite "multi-generation" set $\Lambda$ of frequencies such that:
- $\Lambda$ is invariant for the dynamics of the resonant normal form (i.e. there does not exist a resonance with all the frequencies in $\Lambda$ except one)
- the normal form dynamics restricted to $\Lambda$ contains an orbit undergoing the desired Sobolev norm growth (but is still simple enough to be able to study it)
- Approximation lemma: persistence of the diffusing orbit in the full equation

$$H^{(2)} + H^{(2p+2)}_{Res} + H^{(4p+2)}$$

# Invariant subspaces

There are many invariant subsets $U_\Lambda$
if I choose a set $\Lambda = \{j_1, \dots, j_m\}$ generically then $U_\Lambda$ is invariant and the dynamics on $U_\Lambda$ preserves all the actions $|a_{j_i}|^2$.

This well known fact is the basis for constructing stable and unstable quasi-periodic solutions for the NLS

# Invariant subspaces: General NLS

Think of $\{j_1, \dots, j_m\}$ as a point in $(\mathbb{Z}^2)^m \subset (\mathbb{C}^2)^m$

There exists a proper algebraic variety $\mathfrak{A}$ such that for all $\{j_1, \dots, j_m\} \in (\mathbb{Z}^2)^m \setminus \mathfrak{A}$:

$a_n = 0$ for all $n \neq j_1, \dots, j_m$ is an invariant subspace on which the actions $|a_j|^2$ are constants of motion.

# Invariant subspaces: Cubic NLS

In the cubic case:

$$H_{Birk} = \frac{1}{2} \sum_{\substack{n_1,n_2,n_3,n_4 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 - n_4 = 0 \\ |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0}} a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4}$$

first idea: take $\{j_1, \dots, j_m\}$, $j_i \in \mathbb{Z}^2$

$\circ\, j_1$ $\circ\, j_4$

$\circ\, j_2$ $\circ\, j_3$

any three points do not form a right angle

$a_n = 0$ for all $n \neq j_1, \dots, j_m$ is an invariant subspace on which the actions $|a_j|^2$ are constants of motion.

the condition is $(j_1 - j_2, j_3 - j_2) = 0$

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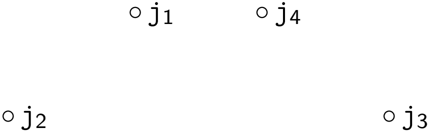


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# Cubic NLS: Integrable dynamics on a rectangle

Consider a rectangle(this is a codimension 3 manifold in $(\mathbb{C}^2)^4$)

$\circ\, \mathtt{j}_1$ $\circ\, \mathtt{j}_4$

$\circ\, \mathtt{j}_3$ $\circ\, \mathtt{j}_2$

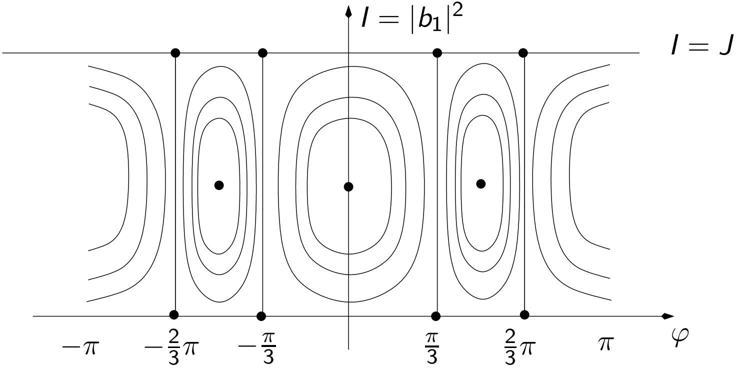
set $a_n = 0$ for all $n \neq j_1, \ldots, j_4$ and $a_{j_1} = a_{j_2} = b_1$, $a_{j_3} = a_{j_4} = b_2$.
(intra-generational equality)
This is an invariant subspace with Hamiltonian

$$h(b_1, b_2) = \frac{1}{4}\left(|b_1|^4 + |b_2|^4\right) - \frac{1}{2}\left(b_1^2 \bar{b}_2^2 + \bar{b}_1^2 b_2^2\right)$$

$$|b_1|^2 + |b_2|^2 = J$$

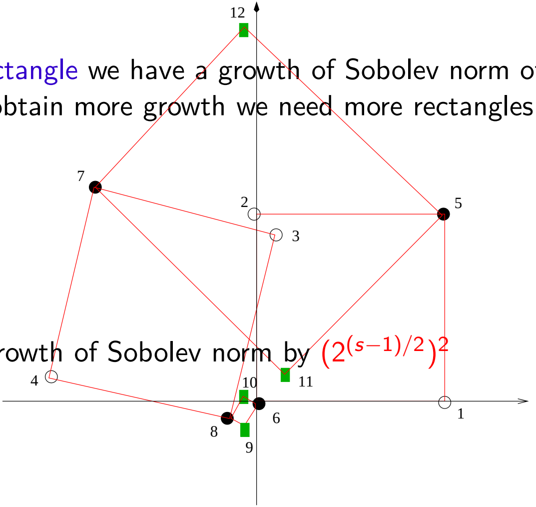
# Integrable dynamics on a rectangle

$$h_{2g} = I_1(J - I_1)\,(1 + 2\cos(2\varphi))\ .$$



# A "genealogical tree" of rectangles: three-generation set $\Lambda$

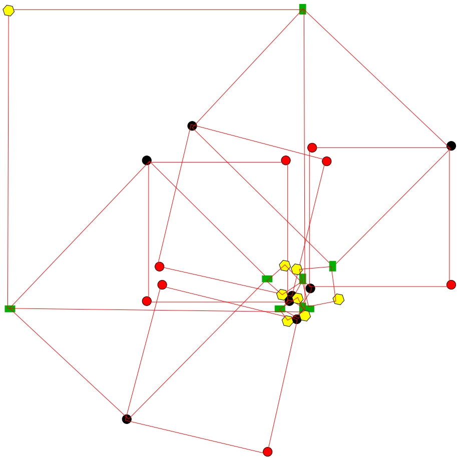
For ONE rectangle we have a growth of Sobolev norm of $2^{(s-1)/2}$
in order to obtain more growth we need more rectangles...




12 points, growth of Sobolev norm by $(2^{(s-1)/2})^2$

# $N = 4$-generations: $N2^{N-1}$ points

growth of Sobolev norm by $(2^{(s-1)/2})^{N-1}$



# The Toy Model: $N$-generations

- Toy model: normal form dynamics restricted to $\Lambda$ a set with $N2^{N-1}$ points.
- Hamiltonian system with $N$ degrees of freedom

$$h(b,\bar{b}) = \frac{1}{4}\sum_{j=1}^{N} |b_j|^4 - \frac{1}{2}\sum_{j=2}^{N}(b_j^2\bar{b}_{j-1}^2 + \bar{b}_j^2 b_{j-1}^2)$$

with symplectic form

$$\Omega = \frac{i}{2} db_j \wedge d\bar{b}_j$$

- conserved quantity $\sum_{j=1}^{N} |b_j|^2 = J$

# Dynamics of the Toy Model

- periodic solutions

$$\mathbb{T}_j = \{b_1 = \ldots = b_{j-1} = b_{j+1} = \ldots = b_N = 0\}$$

- invariant subspaces

$$L_j = \{b_1 = \ldots = b_{j-1} = b_{j+2} = \ldots = b_N = 0\}$$

  on which the system is integrable and contain Heteroclinic connections from $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$

- Local behavior near $\mathbb{T}_j$: two expanding and two contracting directions, all the other directions are elliptic
- existence of a “shadowing” orbit which connects $\mathbb{T}_1$ to $\mathbb{T}_N$ by following closely a sequence of heteroclinic connections

# General NLS

$$-iu_t + \Delta u - |u|^{2p} u = 0$$

$$H_{Birk} = \frac{1}{p+1} \sum_{\substack{n_1,\dots,n_{2p+2}\in\mathbb{Z}^2 \\ \sum(-1)^i n_i = 0 \\ \sum(-1)^i |n_i|^2 = 0}} a_{n_1} \bar{a}_{n_2} \dots a_{n_{2p+1}} \bar{a}_{n_{2p+2}}$$

In principle things might be easier since there are many more resonances

For generic choices of Fourier support $\Lambda = \{j_1, \cdots, j_m\}$ we have periodic solutions supported on $\Lambda$

The hard thing is to find Heteroclinic connections

We have to choose a set $\Lambda$ that is non-generic enough to exhibit non-trivial dynamics (we have to prescribe some resonances), but still is simple enough to allow us to study the dynamics (we have to avoid the appearance of unwanted resonances)

# General NLS: preliminary considerations

Fix ONE resonance for EXAMPLE ($p = 2$, quintic NLS)

$$\Lambda = \{j_1, j_2, j_3, j_4, j_5, j_6\}$$

$$j_1 - j_2 + j_3 - j_4 + j_5 - j_6 = 0\,, \qquad |j_1|^2 - |j_2|^2 + |j_3|^2 - |j_4|^2 + |j_5|^2 - |j_6|^2 = 0$$

Compute the BNF Hamiltonian restricted to this finite set.

The resulting Hamiltonian is integrable.

Question: Does it have heteroclinic connections?

Answer: NO

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$\Lambda$ is a resonance for all NLS of degree $p \geq 2$

$$j_1 - j_2 + j_3 - j_4 + j_5 - j_6 + \underbrace{m - m + \cdots + m - m}_{2p-4} = 0\,,$$

$$|j_1|^2 - |j_2|^2 + |j_3|^2 - |j_4|^2 + |j_5|^2 - |j_6|^2 + \underbrace{|m|^2 - |m|^2 + \cdots + |m|^2 - |m|^2}_{2p-4} = 0$$

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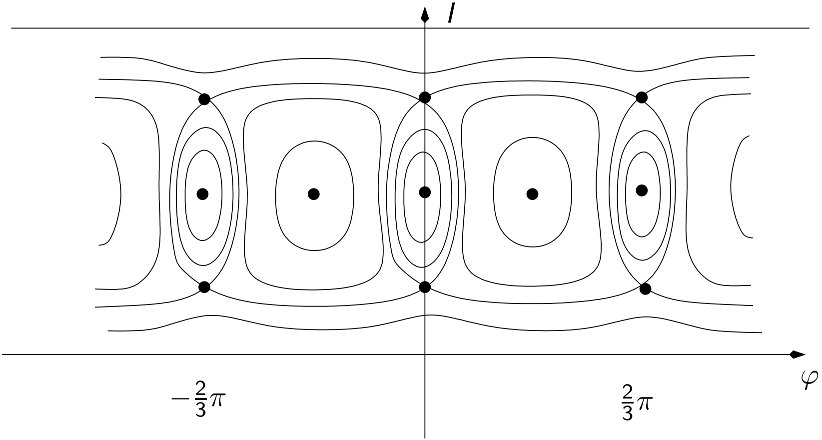
# quintic NLS



# General NLS: preliminary considerations

Fix ANY single resonance

Compute the BNF Hamiltonian restricted to this finite set.

The resulting Hamiltonian is integrable.

Question: Does it have heteroclinic connections?

Answer: The only single resonance which has heteroclinic connection is a rectangle

$$j_1 - j_2 + j_3 - j_4 = 0\,, \quad |j_1|^2 - |j_2|^2 + |j_3|^2 - |j_4|^2 = 0$$

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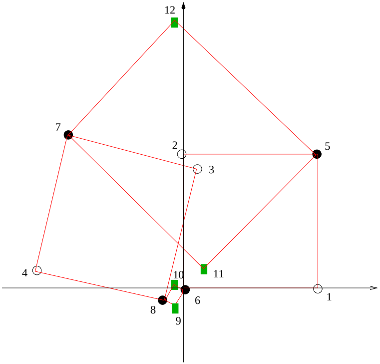
# Avoiding unwanted resonances

Consider the quintic case (the simplest combinatorially non-trivial)
A good starting point is a three generation system




Remember that now a resonance is

$$n_1+n_2+n_3 = n_4+n_5+n_6\,, \quad |n_1|^2+|n_2|^2+|n_3|^2 = |n_4|^2+|n_5|^2+|n_6|^2$$

# Properties of the set $\Lambda$

- One would like to impose the following condition: any nontrivial (i.e. $\{n_1, n_2, n_3\} \neq \{n_4, n_5, n_6\}$) resonant sextuple contained in $\Lambda$ has the form

$$n_1 + n_2 + m = n_3 + n_4 + m$$

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(and the same for the squared moduli)

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So we impose the following:
every point $j_i$ appears in exactly two rectangle resonances.
every nontrivial resonant sextuple contained in $\Lambda$ has

- either the form
$$(n_1, n_2, m, n_3, n_4, m)$$
for some nuclear family $(n_1, n_2, n_3, n_4)$
- or the form
$$(n_1, n_2, n_5, n_3, n_6, n_7)$$
for some nuclear families $(n_1, n_2, n_3, n_4), (n_4, n_5, n_6, n_7)$

The proof of the existence of the frequency set $\Lambda$ requires an accurate and complicated combinatorial analysis that is not needed in the cubic case.

# The Toy Model

$2n = 2^{N-1}$ := cardinality of each generation $\Lambda_i$

$$\frac{3}{n} h(b, \bar{b}) = (6n^2 - 9n + 4) \sum_{k=1}^{N} |b_k|^6 + 9n(2n-1) \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N} |b_k|^4 |b_\ell|^2 +$$
$$+36n^2 \sum_{\substack{k,\ell,m=1 \\ k<\ell<m}}^{N} |b_k|^2 |b_\ell|^2 |b_m|^2 +$$
$$+18 \sum_{k=1}^{N-1} \left( -|b_k|^2 - |b_{k+1}|^2 + n \sum_{\ell=1}^{N} |b_\ell|^2 \right) (b_k^2 \bar{b}_{k+1}^2 + b_{k+1}^2 \bar{b}_k^2) +$$
$$+36 \sum_{k=2}^{N-1} |b_k|^2 (b_{k-1}^2 \bar{b}_{k+1}^2 + b_{k+1}^2 \bar{b}_{k-1}^2) \, .$$

# Combinatorial difficulties

- We use generation sets with rectangles (resonances of the cubic NLS)
- We need to classify which resonances are unavoidable
  as $p$ grows there are more and more unavoidable resonances

The Hamiltonian becomes very complicated but still one can:

- Prove existence of the heteroclinic connections between $\mathbb{T}_j$ and $\mathbb{T}_{j+1}$
- Prove existence of a "slider solution" shadowing the heteroclinic connections from $\mathbb{T}_1$ to $\mathbb{T}_N$
- Prove an *approximation lemma*: a solution with such behavior persists in the full equation (NLS)

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# Flavors of "what might go wrong"

- any integer-coefficient linear combination of family relations is an unavoidable resonance for some $p$ large enough: we PROVE that $\Lambda$ can be constructed so that these are the only resonances
- is $\Lambda$ still invariant for the resonant truncated dynamics?
- does the "intra-generational equality" still define an invariant subspace for the dynamics inside $\Lambda$?
- Dynamics of this complicated toy model?
  - already from the NLS of degree seven, there are always interactions between arbitrarily distant generations
  - because of the increasingly many unavoidable resonances, one cannot give a completely explicit formula for the equations of the toy model
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# general NLS: Toy model

## Theorem

*There exist infinitely many finite dimensional sets Λ, such that the Birkhoff normal form Hamiltonian restricted to this sets is*

$$d!(\sum_{j=1}^{N}|b_j|^2)^{2p-2}(\frac{1}{4}\sum_{j=1}^{N}|b_j|^4-\frac{1}{2}\sum_{j=2}^{N}(b_j^2\bar{b}_{j-1}^2+\bar{b}_j^2 b_{j-1}^2))+O(2^{-N})$$

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this is a small perturbation of the cubic toy model.
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However, IT WORKS!

# Open problems

- 1D non-cubic NLS?
- cubic NLS on more general compact manifolds?
- existence of unbounded Sobolev orbits?

Thanks for your attention!