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University of Zurich[UZH]

A norm approximation to the boson many body dynamics

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joint work with Chiara Boccato and Benjamin Schlein

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Research line: rigorous derivation of time dependent effective equations approximating many body quantum dynamics, in appropriate limiting regimes

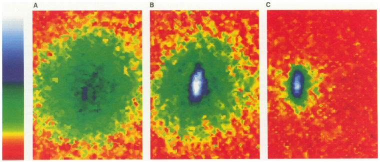
Goals:

- justify the use of the effective equations, which are often introduced on the basis of heuristics or phenomenological arguments
- clarify the limits of applicability of the effective theories, providing bounds on the error of the approximation

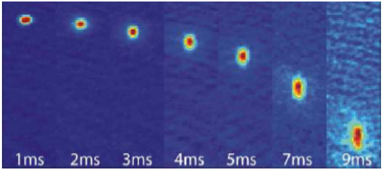
Systems of interest: Bose-Einstein condensates

Statics: describe the appearance of Bose-Einstein condensates in gas of trapped bosons at low temperature.

(→ properties of the ground state of the many-body Hamiltonian)



Anderson et al., BEC in a vapor of Rb-87 (1995)



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Dynamics: after cooling the gas to very low temperatures the traps are switched off, and the evolution of the condensate is observed.

(→ solve the time-dependent many-body Schrdinger equation)

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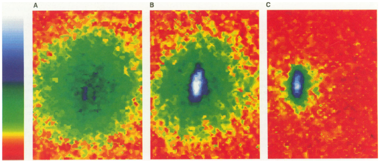

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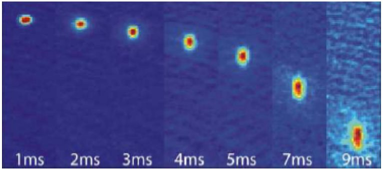

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N bosons in a trap: microscopic description

Wave function $\Psi_{N,0} \in L^2_{sym}(\mathbb{R}^{3N})$ which evolves according to $\Psi_{N,t} = e^{-i H_N t}\Psi_{N,0}$

$$H_N = \sum_{i=1}^{N} \big(-\Delta_{x_i} + V_{ext}(x_j/L) \big) + \frac{a_0}{R_0^3} \sum_{i<j}^{N} V\big((x_i - x_j)/R_0\big)$$

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$V \geq 0$, smooth,
short range

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The scattering length a_0 is defined through the zero energy cross section

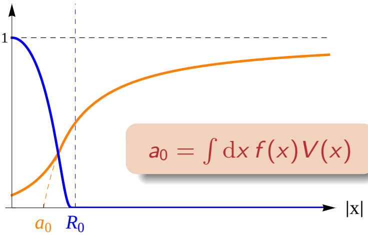
$$\sigma_0 = 4\pi a_0^2 \qquad \text{(for hard sphere potentials: } \sigma_0 = 4\pi R_0^2\text{)}$$

or the zero energy scattering function

$$\big(-\Delta + W/2\big) f = 0\,, \quad f(x) \xrightarrow[|x|\to\infty]{} 1$$

If W is a short range potential

$$f(x) = 1 - \frac{a_0}{|x|} \quad \text{for} \quad |x| > R_0$$



$$a_0 = \int \mathrm{d}x\, f(x) V(x)$$

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Key goal:

- obtain an **approximate macroscopic description** of the system which only takes into account effective interactions in **suitable limiting regimes**

Mean Field limit

A first approximation for the behaviour of Bose condensates: the **mean field scaling limit**

$$H_N^{(MF)} = \sum_{i=1}^{N} \big(-\Delta_{x_i} + V_{ext}(x_j) \big) + \underbrace{\frac{1}{N} \sum_{i<j}^{N} V\big((x_i - x_j)\big)}_{\text{intensity} \sim N^{-1},\ \text{range} \sim 1}$$

The Mean Field potential describes **many weak collisions**. Correlations among the particles are negligible.

The ground state wave function can be approximated for large N by

$$\Psi_N^{(MF)} \simeq \varphi^{\otimes N},$$

with $\varphi \in L^2(\mathbb{R}^3)$ the minimizer of the Hartree functional:

$$\mathcal{E}_{MF}(\varphi) = \int \mathrm{d}x \Big(|\nabla \varphi(x)|^2 + V_{ext}(x)|\varphi(x)|^2 + (V \star |\varphi|^2)(x)|\varphi(x)|^2 \Big)$$

Gross-Pitaevskii scaling limit

A better approximation: the **Gross–Pitaevskii scaling limit**

$$H_N^{(GP)} = \sum_{i=1}^{N} \left(-\Delta_{x_i} + V_{ext}(x_j) \right) + \underbrace{\sum_{i<j}^{N} N^2 V\left(N(x_i - x_j)\right)}_{\text{intensity}\sim N^2,\ \text{range}\sim N^{-1}}$$

Gross-Pitaevskii potential describes **rare but strong collisions**:

- Fix one particle, probability of a collision: $\rho R_0^3 = N \cdot N^{-3} = N^{-2}$
- Average number of collisions: N^{-1}

The potential $N^2 V(Nx)$ has scattering function $f(Nx) = 1 - \frac{a_0}{N|x|}$ and scattering length $a = a_0/N$

Physically relevant: in typical condensates $N \simeq 10^3 - 10^6$ and a is such that $Na \sim 1$.

Gross-Pitaevskii: ground state properties (1)

Correlations among the particles play a crucial role: the ground state energy of the boson gas depends on the scattering length of the interaction potential.

[Lieb-Seiringer-Yngvason, 2000] Let E_N be the ground state energy of

$$H_N^{(GP)} = \sum_{i=1}^{N} \left(-\Delta_{x_i} + V_{ext}(x_j) \right) + \sum_{i<j}^{N} N^2 V\left(N(x_i - x_j)\right)$$

Then

$$E_N/N \xrightarrow[N\to\infty]{} \min_{\varphi \in L^2(\mathbb{R}^3): \|\varphi\|=1} \mathcal{E}_{GP}(\varphi) = \mathcal{E}_{GP}(\varphi_{GP})$$

with $\mathcal{E}_{GP}(\varphi)$ the Gross-Pitaevskii energy functional

$$\mathcal{E}_{GP}(\varphi) = \int \mathrm{d}x \left(|\nabla \varphi(x)|^2 + V_{ext}(x)|\varphi(x)|^2 + 4\pi a_0 |\varphi(x)|^4 \right)$$

Gross-Pitaevskii: ground state properties (2)

[Lieb-Seiringer, 2002] Let $\Psi_N(x_1, \dots, x_N)$ the ground state of

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and $\gamma_N^{(1)}$ be the one particle reduced density associated to $\Psi_N(x_1, \dots, x_N)$

$$\gamma_N^{(1)} := \mathrm{Tr}_{2,\dots,N} |\Psi_N\rangle\langle\Psi_N|$$

Then, in trace norm

$$\gamma_N^{(1)} \xrightarrow[N\to\infty]{} |\varphi_{GP}\rangle\langle\varphi_{GP}|$$

where $\varphi_{GP} \in L^2(\mathbb{R}^3)$ the minimizer of the GP functional $\mathcal{E}_{GP}(\varphi)$. This also implies:

$$\gamma_N^{(k)} \xrightarrow[N\to\infty]{} |\varphi_{GP}\rangle\langle\varphi_{GP}|^{\otimes k}$$

Complete condensation in the g.s.: the expectation of any k-particle observable in the ground state can be computed using $\varphi_{GP} \otimes \cdots \otimes \varphi_{GP}$ instead than Ψ_N.

Gross-Pitaevskii: emergence of correlations

Complete condensation in the GP regime does NOT mean that $\Psi_N \simeq \varphi^{\otimes N}$

$\langle \varphi^{\otimes N}, H_N^{(GP)} \varphi^{\otimes N} \rangle$ (Energy of a factorized state)

$$= N\Big[\int \mathrm{d}x \left(|\nabla \varphi(x)|^2 + V_{ext} |\varphi(x)|^2 \right) + \frac{1}{2} \int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y)) |\varphi(x)|^2 |\varphi(y)|^2 \Big]$$

$$\xrightarrow[N \to \infty]{} \frac{b_0}{2} \int \mathrm{d}x\mathrm{d}y \delta(x-y) |\varphi(x)|^2 |\varphi(y)|^2$$

$$\simeq N \int \mathrm{d}x \left(|\nabla \varphi(x)|^2 + V_{ext} |\varphi(x)|^2 + \frac{b_0}{2} |\varphi(x)|^4 \right)$$

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Ground state energy in the Gross-Pitaevskii regime:

$$E_N / N \xrightarrow[N\to\infty]{} \int \mathrm{d}x \left(|\nabla \varphi(x)|^2 + V_{ext} |\varphi(x)|^2 + 4\pi a_0 |\varphi(x)|^4 \right)$$

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Correlations generated by the strong collisions disappear in $\gamma_N^{(k)}$, but crucially affect the ground state energy.

Between MF and GP scaling limits

Let $0 < \beta < 1$. Intermediate scaling limits between MF and GP are:

$$H_N^{(\beta)} = \sum_{i=1}^{N} \big(-\Delta_{x_i} + V_{ext}(x_j) \big) + \underbrace{\frac{1}{N} \sum_{i<j}^{N} N^{3\beta} V\big(N^{\beta}(x_i - x_j)\big)}_{\text{intensity}\sim N^{3\beta-1},\ \text{range}\sim N^{-\beta}}$$

For any $0 < \beta < 1$

- the potential $N^{3\beta-1} V(N^{\beta} x)$ has scattering length $a \simeq N^{-1}$ *i.e.* particles are correlated up to distances N^{-1}
- the range of the interaction $R_0 \simeq N^{-\beta}$ is much larger than a

Key point: correlations among particles are not negligible, but since $a \ll R_0$ they affect the ground state energy at lowest order with respect to the Gross-Pitaevskii scaling limit.

Dynamics: known results (3d)

$$H_N^{(\beta)} = \sum_{i=1}^N -\Delta_{x_i} + \sum_{i<j}^N V_N^{(\beta)}(x_i - x_j)$$

	$V_N^{(\beta)}(x)$	Effective equation		
Mean Field ($\beta = 0$)	$\frac{1}{N} V(x)$	$i\partial_t \varphi_t = -\Delta\varphi_t + (V \star \lvert\varphi_t\rvert^2)\varphi_t$		
$\beta \in (0,1)$	$N^{3\beta-1} \cdot V(N^\beta x)$	$i\partial_t \varphi_t = -\Delta\varphi_t + (\int V)\lvert\varphi_t\rvert^2\varphi_t$		
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Trace norm convergence: let the initial state Ψ_N be condensate and let $\gamma_{N,t}^{(1)}$ be the reduced density matrix associated to $\Psi_{N,t} = e^{-i H_N^{(\beta)} t}\Psi_N$. Then, $\gamma_{N,t}^{(1)} \to \lvert\varphi_t\rangle\langle\varphi_t\rvert$, where φ_t solves the corresponding effective equation.

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$$\left\|\Psi_{N,t} - \Psi_{approx,t}\right\| \to 0\,,$$

where $\Psi_{approx,t}$ depends only on a few degrees of freedom (including φ_t).

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Gross-Pitaevskii ($\beta = 1$)	$N^2 V(Nx)$	$i\partial_t \varphi_t = -\Delta \varphi_t + 8\pi a_0 \lvert\varphi_t\rvert^2 \varphi_t$	Erdös-Schlein-Yau, Pickl, Benedikter et al.	

Norm approximation: let $\Psi_{N,t} = e^{-i H_N^{(\beta)} t} \Psi_N$. Then,

$$\left\| \Psi_{N,t} - \Psi_{approx,t} \right\| \to 0\,,$$

where $\Psi_{approx,t}$ depends only on a few degrees of freedom (including φ_t).

Dynamics: known results (3d)

$$H_N^{(\beta)} = \sum_{i=1}^N -\Delta_{x_i} + \sum_{i<j}^N V_N^{(\beta)}(x_i - x_j)$$

	$V_N^{(\beta)}(x)$	Effective equation	Trace Norm	Norm approx
Mean Field ($\beta = 0$)	$\frac{1}{N} V(x)$	$i\partial_t \varphi_t = -\Delta\varphi_t + (V \star \lvert\varphi_t\rvert^2)\varphi_t$	Hepp, Ginibre-Velo, Spohn, Erdös-Yau, Rodnianskii-Schlein, Knowles-Pickl	Ginibre-Velo, Grillakis et al., Ben Arous et al.
$\beta \in (0,1)$	$N^{3\beta-1} \cdot V(N^\beta x)$	$i\partial_t \varphi_t = -\Delta\varphi_t + (\int V)\lvert\varphi_t\rvert^2\varphi_t$	Erdös-Schlein-Yau Pickl	Grillakis-M. ($\beta < 1/3$) **?**
Gross-Pitaevskii ($\beta = 1$)	$N^2 V(Nx)$	$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi a_0 \lvert\varphi_t\rvert^2\varphi_t$	Erdös-Schlein-Yau, Pickl, Benedikter et al.	**?**

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Mean Field case. Fluctuations around Hartree dynamics satisfy a CLT

$$\frac{1}{\sqrt{N}} \sum_{i=1}^N \mathcal{O}_i^{(1)} - \langle \varphi_t, \mathcal{O}^{(1)} \varphi_t \rangle \;\to\; \text{Gauss}(0, \sigma_t^2)$$

and Bogoliubov prediction for the spectrum holds (Seiringer-Grech, Lewin et al.)

Dynamics: known results (3d)

$$H_N^{(\beta)} = \sum_{i=1}^N -\Delta_{x_i} + \sum_{i<j}^N V_N^{(\beta)}(x_i - x_j)$$

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Mean Field ($\beta = 0$)	$\frac{1}{N} V(x)$	$i\partial_t \varphi_t = -\Delta \varphi_t + (V \star \lvert\varphi_t\rvert^2)\varphi_t$	Hepp, Ginibre-Velo, Spohn, Erdös-Yau, Rodnianskii-Schlein, Knowles-Pickl	Ginibre-Velo, Grillakis et al., Ben Arous et al.
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Strategy: apply the method developed by Rodianski-Schlein, based on

- a representation of the many boson system on the **Fock space**
- the study of the time evolution of **coherent states**

Fock space

Idea: **enlarge the set of initial states**, embedding $L^2_s(\mathbb{R}^{3N})$ in the bosonic Fock space:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} L^2_s(\mathbb{R}^{3n}) = \mathbb{C} \oplus L^2_s(\mathbb{R}^3) \oplus \cdots \oplus L^2_s(\mathbb{R}^N) \oplus \cdots$$

$$\psi = \{\psi^{(0)}, \ \psi^{(1)} \ , \ \ldots \ , \ \psi^{(N)} \ , \ \ldots \ \} \in \mathcal{F}$$

- $\|\psi^{(n)}\|^2 =$ prob. of having n particles in the state ψ ($\sum_n \|\psi^{(n)}\|^2 = 1$)
- vacuum state: $\Omega = \{1, 0, 0, \ldots, 0\}$
- state with fixed particle number: $\{0, 0, \ldots, \psi_N, \ldots, 0\}$

Creation and annihilation operators

Creation and annihilation operators: for $f \in L^2(\mathbb{R}^3)$ define

$$a^*(f) : \mathcal{H}^{(n)} \to \mathcal{H}^{(n+1)}, \qquad a(f) : \mathcal{H}^{(n+1)} \to \mathcal{H}^{(n)}$$

whose action is given by

$$\big(a^*(f)\psi\big)^{(n+1)}(x_1, \dots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} f(x_j)\, \psi^{(n)}(x_1, \dots, \not{x}_j, \dots, x_{n+1})$$

$$\big(a(f)\psi\big)^{(n)}(x_1, \dots, x_n) = \sqrt{n+1} \int \mathrm{d}x\, f(x)\, \psi^{(n+1)}(x_1, \dots, x_n, x)$$

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Canonical commutation relations:

$$[a(f), a^*(g)] = \langle f, g \rangle_{L^2}, \qquad [a(f), a(g)] = [a^*(f), a^*(g)] = 0$$

Action of a and a^*:

$$a(\varphi)\Omega = 0; \qquad \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega = \{0, \ldots, \varphi^{\otimes N}, \ldots, 0\}$$

Fock space Hamiltonian

Operator valued distributions a_x and a_x^* :

$$a^*(f) = \int f(x)\, a_x^* \mathrm{d}x \,; \qquad a(f) = \int \overline{f(x)}\, a_x \mathrm{d}x$$

Number of particle operator: $\mathcal{N} = \int \mathrm{d}x\, a_x^* a_x$

Fock space Gross-Pitaevskii Hamiltonian:

$$\mathcal{H}_N = \int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \frac{1}{2} \int \mathrm{d}x \mathrm{d}y\, N^2 V(N(x-y)) a_x^* a_y^* a_y a_x$$

Note that

$$\left(\mathcal{H}_N \psi\right)^{(m)} = H_N^{(m)} \psi^{(m)}$$

with

$$H_N^{(m)} = \sum_{j=1}^{m} -\Delta_{x_j} + N^2 \sum_{i<j}^{m} V(N(x_i - x_j))$$

Coherent states

For $\varphi \in L^2(\mathbb{R}^3)$ define the **Weyl operator**

$$W(\varphi) = e^{a^*(\varphi) - a(\varphi)}$$

Coherent state with wave function φ :

$$W(\varphi)\Omega = e^{-\|\varphi\|^2/2}\left\{1,\ \varphi,\ \frac{\varphi^{\otimes 2}}{\sqrt{2!}},\ \frac{\varphi^{\otimes 3}}{\sqrt{3!}}, \dots, \frac{\varphi^{\otimes N}}{\sqrt{N!}}, \dots\right\}$$

- Coherent states are normalized: $W^*(\varphi) = W(\varphi)^{-1}$
- Action of the Weyl operator:

$$W^*(\varphi) a_x W(\varphi) = a_x + \varphi(x) \quad \Rightarrow \quad a_x W(\varphi)\Omega = \varphi(x) W(\varphi)\Omega$$
$$W^*(\varphi) a_x^* W(\varphi) = a_x^* + \overline{\varphi(x)}$$

- Expected number of particle: $\langle W(\varphi)\Omega, \mathcal{N} W(\varphi)\Omega\rangle = \|\varphi\|_2^2$

Coherent states

Initial state with average particle number N:

$\Psi_{N,0} = W(\sqrt{N}\varphi)\Omega$

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Gross-Pitaevskii regime: coherent state approach

Gross-Pitaevskii Fock space Hamiltonian:

$$\mathcal{H}_N^{(GP)} = \int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \frac{1}{2} \int \mathrm{d}x\mathrm{d}y\, N^2 V(N(x-y)) a_x^* a_y^* a_y a_x$$

First idea: we assume the evolution of a coherent state to be approximately coherent with φ_t solution of GP equation:

$$e^{-it\mathcal{H}_N^{GP}} W(\sqrt{N}\varphi)\Omega \simeq W(\sqrt{N}\varphi_t)\Omega$$

Then

$$\begin{aligned} N\gamma_{N,t}^{(1)}(x,y) &= \left\langle e^{-i\mathcal{H}_N^{MF} t} W(\sqrt{N}\varphi)\Omega, a_y^* a_x\, e^{-i\mathcal{H}_N^{MF} t} W(\sqrt{N}\varphi)\Omega \right\rangle \\ &\simeq \left\langle W(\sqrt{N}\varphi_t)\Omega, a_y^* a_x\, W(\sqrt{N}\varphi_t)\Omega \right\rangle \\ &= \left\langle \Omega, (\, a_y^* + \sqrt{N}\,\overline{\varphi_t(y)}\,)(\, a_x + \sqrt{N}\,\varphi_t(x)\,)\,\Omega \right\rangle \\ &= N\, \overline{\varphi_t(y)}\, \varphi_t(x) \end{aligned}$$

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Rigorous approach: define

$$e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi)\Omega := W(\sqrt{N}\varphi_t)\mathcal{U}_N(t)\Omega$$

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$$W(\sqrt{N}\varphi_t) W^*(\sqrt{N}\varphi_t) e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi)\Omega := W(\sqrt{N}\varphi_t)\mathcal{U}_N(t)\Omega$$

where

$$\mathcal{U}_N(t) = W^*(\sqrt{N}\varphi_t)\, e^{-it\mathcal{H}_N^{(GP)}}\, W(\sqrt{N}\varphi)$$

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where

$$\mathcal{U}_N(t) = W^*(\sqrt{N}\varphi_t)\, e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi)$$

One obtain:

$$\mathrm{Tr}\left|\gamma_{N,t}^{(1)}(x,y) - |\varphi_t\rangle\langle\varphi_t|\right| \le \frac{C}{\sqrt{N}} \langle \mathcal{U}_N(t)\Omega, \mathcal{N}\mathcal{U}_N(t)\Omega\rangle$$

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One obtain:

$$\mathrm{Tr}\left|\gamma_{N,t}^{(1)}(x,y) - |\varphi_t\rangle\langle\varphi_t|\right| \leq \frac{C}{\sqrt{N}} \langle \mathcal{U}_N(t)\Omega, \mathcal{N}\mathcal{U}_N(t)\Omega\rangle$$

The problem reduces to **control of the number of fluctuations**: is it possible to show that

$$\langle \mathcal{U}_N(t)\Omega, \mathcal{N}\mathcal{U}_N(t)\rangle \leq D(t) \qquad ?$$

Gross-Pitaevskii regime: coherent state approach

$$\frac{d}{dt}\langle \mathcal{U}_N(t)\Omega, \mathcal{N}\mathcal{U}_N(t)\Omega\rangle \leq |\langle \mathcal{U}_N(t)\Omega, [\mathcal{L}_N(t), \mathcal{N}]\mathcal{U}_N(t)\Omega\rangle|$$

with $i\partial_t \mathcal{U}_N(t) = \mathcal{L}_N(t)\mathcal{U}_N(t)$

$$\begin{aligned}
\mathcal{L}_N(t) = &\int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \frac{1}{N}\int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y)) a_x^* a_y^* a_x a_y \\
&+ \sqrt{N}\int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y))(1-f(N(x-y)))\left[|\varphi_t^{(N)}(y)|^2 \varphi_t^{(N)}(x) a_x^* + \mathrm{h.c.}\right] \\
&+ \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y))\left(\varphi^{(N)}(x)\varphi^{(N)}(y) a_y^* a_x^* + \mathrm{h.c.}\right) \\
&+ \int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y))\left(|\varphi_t^{(N)}(x)|^2 a_y^* a_y + \varphi_t^{(\bar{N})}(x)\varphi_t^{(N)}(y) a_y^* a_x\right) \\
&+ \frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y))\left(\overline{\varphi_t^{(N)}(y)}\, a_x^* a_x a_y + \mathrm{h.c.}\right)
\end{aligned}$$

Gross-Pitaevskii regime: coherent state approach

$$\frac{d}{dt}\langle \mathcal{U}_N(t)\Omega, \mathcal{N}\mathcal{U}_N(t)\Omega\rangle \leq |\langle \mathcal{U}_N(t)\Omega, [\mathcal{L}_N(t), \mathcal{N}]\mathcal{U}_N(t)\Omega\rangle| \sim \sqrt{N}$$

with $i\partial_t \mathcal{U}_N(t) = \mathcal{L}_N(t)\mathcal{U}_N(t)$

$$\begin{aligned}\mathcal{L}_N(t) = &\int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \frac{1}{N}\int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y)) a_x^* a_y^* a_x a_y \\ &+\sqrt{N}\int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y))(1-f(N(x-y)))\left[|\varphi_t^{(N)}(y)|^2 \varphi_t^{(N)}(x) a_x^* + \mathrm{h.c.}\right] \\ &+\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y))\left(\varphi^{(N)}(x)\varphi^{(N)}(y) a_y^* a_x^* + \mathrm{h.c.}\right) \\ &+\int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y))\left(|\varphi_t^{(N)}(x)|^2 a_y^* a_y + \varphi_t^{(\bar{N})}(x)\varphi_t^{(N)}(y) a_y^* a_x\right) \\ &+\frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y))\left(\overline{\varphi_t^{(N)}(y)}\, a_x^* a_x a_y + \mathrm{h.c.}\right)\end{aligned}$$

Heuristically, $e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi)\Omega$ **develops singular correlations**, which are not captured by the approximate evolution $W(\sqrt{N}\varphi_t)\Omega$.

Gross-Pitaevskii regime: squeezed coherent states

- Define a **correlation structure**:

$$k_t(x,y) = -N\big(1 - f(N(x-y))\big)\varphi_t(x)\varphi_t(y) \qquad (\|k_t\|_{L^2\times L^2} \leq C)$$

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- **Implement correlations** using a Bogoliubov transformation

$$T(k_t) := e^{\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\left(k_t(x,y)a_x^* a_y^* - \bar{k}_t(x,y)a_x a_y\right)}$$

whose action is

$$T^*(k_t)\,a(f)\,T(k_t) = a(\cosh_{k_t}(f)) + a^*(\sinh_{k_t}(\bar{f})) \simeq a(f) + a^*(k_t\bar{f})$$
$$T^*(k_t)\,a^*(f)\,T(k_t) = a^*(\cosh_{k_t}(f)) + a(\sinh_{k_t}(\bar{f})) \simeq a^*(f) + a(k_t\bar{f})$$

with $\cosh_{k_t} = \sum_{j=0}^{\infty} \frac{(k_t\bar{k}_t)^n}{2n!}$ and $\sinh_{k_t} = k_t + \sum_{j=1}^{\infty} \frac{(k_t\bar{k}_t)^n k_t}{(2n+1)!}$

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- **Add short-scale correlations.** New initial data:

$$\psi_{N,0} = W(\sqrt{N}\varphi)\,T(k_0)\,\Omega \qquad \text{(squeezed coherent state)}$$

Gross-Pitaevskii regime: modified fluctuation dynamics

Gross-Pitaevskii Hamiltonian:

$$\mathcal{H}_N^{(GP)} = \int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \frac{1}{2} \int \mathrm{d}x\mathrm{d}y\, N^2 V(N(x-y)) a_x^* a_y^* a_y a_x$$

We assume

$$e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi) T(k_0) \Omega \simeq W(\sqrt{N}\varphi_t)\, T(k_t)\, \Omega$$

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We define **a new fluctuation operator** $\tilde{\mathcal{U}}_N(t)$:

$$e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi) T(k_0)\,\Omega := W(\sqrt{N}\varphi_t)\, T(k_t) \tilde{\mathcal{U}}_N(t)\Omega$$

where

$$\tilde{\mathcal{U}}_N(t) = T^*(k_t) W^*(\sqrt{N}\varphi_t) e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi) T(k_0)$$

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where

$$\tilde{\mathcal{U}}_N(t) = T^*(k_t) W^*(\sqrt{N}\varphi_t) e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi) T(k_0)$$

As before:

$$\mathrm{Tr}\left|\gamma_{N,t}^{(1)}(x,y) - |\varphi_t\rangle\langle\varphi_t|\right| \leq \frac{C}{\sqrt{N}} \langle \tilde{\mathcal{U}}_N(t)\Omega, \mathcal{N}\tilde{\mathcal{U}}_N(t)\Omega\rangle$$

The number of fluctuations w.r.t. the new dynamics stays of order one

$$\langle \tilde{\mathcal{U}}_N(t)\Omega, \mathcal{N}\tilde{\mathcal{U}}_N(t)\rangle \leq e^{c_1 \exp(c_2)|t|}$$

Gross-Pitaevskii regime: modified fluctuation dynamics

The generator of $\tilde{\mathcal{U}}_N(t) = T^*(k_t)W^*(\sqrt{N}\varphi_t)e^{-i\mathcal{H}_N^{(GP)}t}W(\sqrt{N}\varphi)T(k_0)$ is:

$$\begin{aligned}
\tilde{\mathcal{L}}_N(t) &= [i\partial_t T^*(k_t)]T(k_t) \\
&+ T^*(k_t)\Big[\int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \frac{1}{N}\int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y)) a_x^* a_y^* a_x a_y \\
&+\sqrt{N}\int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y))(1-f(N(x-y)))\Big[|\varphi_t^{(N)}(y)|^2 \varphi_t^{(N)}(x) a_x^* + \text{h.c.}\Big] \\
&+ \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y)) \left(\varphi^{(N)}(x)\varphi^{(N)}(y) a_y^* a_x^* + \text{h.c.}\right) \\
&+ \int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y)) \left(|\varphi_t^{(N)}(x)|^2 a_y^* a_y + \varphi_t^{(\bar{N})}(x)\varphi_t^{(N)}(y) a_y^* a_x\right) \\
&+ \frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y)) \left(\overline{\varphi_t^{(N)}(y)}\, a_x^* a_x a_y + \text{h.c.}\right)\Big] T(k_t)
\end{aligned}$$

Gross-Pitaevskii regime: modified fluctuation dynamics

The generator of $\tilde{\mathcal{U}}_N(t) = T^*(k_t)W^*(\sqrt{N}\varphi_t)e^{-i\mathcal{H}_N^{(GP)}t}W(\sqrt{N}\varphi)T(k_0)$ is:

$$
\begin{aligned}
\tilde{\mathcal{L}}_N(t) &= [i\partial_t T^*(k_t)]T(k_t) \\
&+ T^*(k_t)\Big[\int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \frac{1}{N}\int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y)) a_x^* a_y^* a_x a_y \\
&+\sqrt{N}\int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y))(1-f(N(x-y)))\Big[|\varphi_t^{(N)}(y)|^2\varphi_t^{(N)}(x)a_x^* + \mathrm{h.c.}\Big] \\
&+ \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y))\left(\varphi^{(N)}(x)\varphi^{(N)}(y)a_y^* a_x^* + \mathrm{h.c.}\right) \\
&+ \int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y))\left(|\varphi_t^{(N)}(x)|^2 a_y^* a_y + \varphi_t^{(\bar{N})}(x)\varphi_t^{(N)}(y)a_y^* a_x\right) \\
&+ \frac{1}{\sqrt{N}}\int \mathrm{d}x\mathrm{d}y\, N^3 V(N(x-y))\left(\overline{\varphi_t^{(N)}(y)}\, a_x^* a_x a_y + \mathrm{h.c.}\right)\Big] T(k_t)
\end{aligned}
$$

Key point. The action of Bogoliubov transformation destroys normalorder:

$$
\begin{aligned}
T^*(k_t)\big(\sqrt{N}a^\# + \tfrac{1}{\sqrt{N}}a^\# a^\# a^\#\big)T(k_t) &= \sqrt{N}(\text{linear}) + \tfrac{1}{\sqrt{N}}(\text{cubic, not normalorder}) \\
&= \sqrt{N}(\text{linear}) - \sqrt{N}(\text{linear}) + \tfrac{1}{\sqrt{N}}(\text{cubic, normalorder})
\end{aligned}
$$

Gross-Pitaevskii regime: rate of convergence

Theorem [Benedikter-de Oliveira-Schlein, 2012] Let $\|\varphi\|_{L^2(\mathbb{R}^3)} = 1$, $V \geq 0$ and

$$\mathcal{H}_N^{(GP)} = \int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \frac{1}{2} \int \mathrm{d}x\mathrm{d}y\, N^2 V(N(x-y)) a_x^* a_y^* a_y a_x$$

- Let $\gamma_{N,t}^{(1)}$ be the reduced density of

$$\Psi_{N,t} = e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi) T(k_0)\Omega$$

- Let φ_t solve the Gross-Pitaevskii equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t\,, \qquad \text{with initial data } \varphi_0 = \varphi$$

Then

$$\mathrm{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq C N^{-1/2} e^{c_1 \exp(c_2|t|)}$$

- The class of initial data $\Psi_{N,0} = W(\sqrt{N}\varphi)T(k_0)\Omega$ has expected particle number N and "correct energy": $\left\langle \Psi_{N,0}, \mathcal{H}_N^{(GP)} \Psi_{N,0} \right\rangle = N\mathcal{E}_{GP}(\varphi) + \mathcal{O}(1)$
- The same strategy can be applied for any $0 < \beta < 1$.

A norm approximation result

[Boccato-C.-Schlein, 2015] Let $0 < \beta < 1$

$$\mathcal{H}_N^{(\beta)} = \int \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + N^{3\beta-1} \int \mathrm{d}x\mathrm{d}y\, V(N^\beta(x-y)) a_x^* a_y^* a_y a_x$$

with V smooth, positive, spherically symmetric and with compact support, and $T(k_t) = \exp(\int k_t(x,y) a_x^* a_y^* - \mathrm{h.c.})$ with correlation kernel

$$k_t(x,y) = -N\left(1 - f_{\ell,N}(x-y)\right) \varphi_t^2((x+y)/2)$$

with a "modified" scattering function $f_{\ell,N}$ satisfying

$$\begin{cases} \left(-\Delta + \frac{1}{2} N^{3\beta-1} V(N^\beta x)\right) f_{\ell,N}(x) = \lambda_{\ell,N} f_{\ell,N}(x)\,, & |x| \le \ell \\ f_{\ell,N}(\ell) = 1\,,\ \partial_r f_{\ell,N}(\ell) = 0 \end{cases}$$

and φ_t solution of $i\partial_t \varphi_t = -\Delta\varphi_t + (\int V)|\varphi_t|^2 \varphi_t$. Then

$$\left\| \left(e^{-it\mathcal{H}_N^{(\beta)}} W(\sqrt{N}\varphi) T(k_0)\Omega - W(\sqrt{N}\varphi_t)\, T(k_t) \mathcal{U}_2 \Omega \right\| \xrightarrow[N\to\infty]{} 0$$

Remarks

$$\|(e^{-it\mathcal{H}_N^{(\beta)}}W(\sqrt{N}\varphi)T(k_0)\Omega - W(\sqrt{N}\varphi_t)\,T(k_t)\mathcal{U}_2\Omega)\| \leq C\,N^{-\gamma}e^{c_1\exp(c_2|t|)}$$

$$\gamma = \min(\beta/4, (1-\beta)/4)$$

Limit dynamics. $\mathcal{U}_2$ has a quadratic generator, as a consequence it depends only on few parameters ($\rightarrow$ great simplification)

Initial data. The theorem also holds for initial data of the form $W(\sqrt{N}\varphi)T(k_0)\psi$ with $\psi \in \mathcal{F}$ satisfying

$$\langle\psi, (\mathcal{N}^2 + \mathcal{K}^2 + \mathcal{H}_N^{(\beta)})\psi\rangle \leq C$$

Comparison with Mean Field. Similar result without introducing correlations:

$$\|(e^{-it\mathcal{H}_N^{(MF)}}W(\sqrt{N}\varphi)\Omega - W(\sqrt{N}\varphi_t)\,\mathcal{U}_2^{(MF)}\Omega)\| \leq C\,N^{-1/2}e^{k|t|}$$

Strategy of the proof

Our goal si to prove that:

$$\| \underbrace{e^{-it\mathcal{H}_N^{(\beta)}} W(\sqrt{N}\varphi) T(k_0)\Omega}_{W(\sqrt{N}\varphi_t) T(k_t) \tilde{\mathcal{U}}_N(t)\Omega} - W(\sqrt{N}\varphi_t) T(k_t) \mathcal{U}_2(t)\Omega \|^2 \leq \frac{C(t)}{N^\gamma}$$

It is sufficient to show that

$$\|(\tilde{\mathcal{U}}_N(t) - \mathcal{U}_2(t))\Omega\|^2 \leq \frac{C(t)}{N^\gamma}$$

One uses

$$\frac{\mathrm{d}}{\mathrm{d}t} \|(\tilde{\mathcal{U}}_N(t) - \mathcal{U}_2(t))\Omega\|^2 = \mathrm{Im}\, \langle \tilde{\mathcal{U}}_N \Omega, (\tilde{\mathcal{L}}_N - \mathcal{L}_2)\mathcal{U}_2\Omega \rangle$$

Goal: show that $|\langle \tilde{\mathcal{U}}_N \Omega, (\tilde{\mathcal{L}}_N - \mathcal{L}_2)\mathcal{U}_2\Omega \rangle| \leq C(t) N^{-\gamma}$

Strategy of the proof

One finds:

$$|\langle \tilde{\mathcal{U}}_N \Omega, (\tilde{\mathcal{L}}_N - \mathcal{L}_2)\mathcal{U}_2 \Omega \rangle| \leq \frac{C(t)}{N^{-\delta}} \Big[\langle \tilde{\mathcal{U}}_N \psi, \mathcal{V}_N + \mathcal{K} + \mathcal{N} + 1, \tilde{\mathcal{U}}_N \psi \rangle + \langle \mathcal{U}_2 \psi, (\mathcal{K} + \mathcal{N} + 1)^2, \mathcal{U}_2 \psi \rangle \Big]$$

The problem ends up in

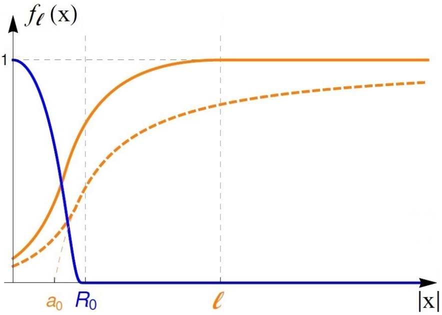
- controlling the growth of $\mathcal{N}$, $\mathcal{V}_N$ and $\mathcal{K}$ w.r.t. the full dynamics $\tilde{\mathcal{U}}_N$: similar to what done in [Benedikter- de Oliveira-Schlein, 12]
- controlling the growth of $\mathcal{N}^2$ and $\mathcal{K}^2$ w.r.t. the limiting dynamics $\mathcal{U}_2$: this step requires a careful choice of the correlation kernel

Correlation structure

We denote with $f_{\ell,N}$ the ground state solution of the Neumann problem:

$$\begin{cases} \left(-\Delta + \frac{1}{2}N^{3\beta-1}V(N^\beta x)\right) f_{\ell,N}(x) \\ \qquad\qquad = \lambda_{\ell,\beta} f_{\ell,N}(x)\,, \quad |x| \leq \ell \\ f_{\ell,N}(\ell) = 1\,, \quad \partial_r f_{\ell,N}(\ell) = 0 \end{cases}$$

with $a_0 \sim N^{-1}$, $R_0 \sim N^{-\beta}$ and $\ell \sim 1$.



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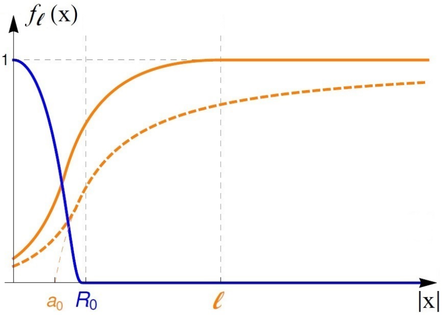

Idea: particles are correlated up to distances of order 1.

$$k_t(x,y) = -N\left(1 - f_{\ell,N}(x-y)\right)\varphi_t^2(x+y/2)$$

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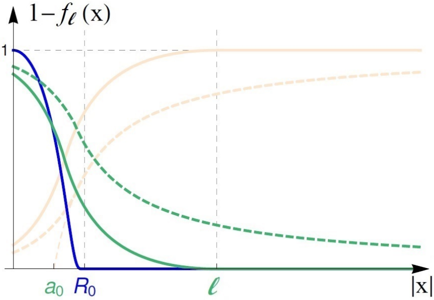

Idea: particles are correlated up to distances of order 1.

$$k_t(x,y) = -N\left(1 - f_{\ell,N}(x-y)\right)\varphi_t^2({}^{x+y}/_2)$$

Extension to the Gross-Pitaevskii regime

Goal: show that $|\langle \tilde{\mathcal{U}}_N\Omega, (\tilde{\mathcal{L}}_N(t) - \mathcal{L}_2(t))\mathcal{U}_2\Omega\rangle| \le C(t)N^{-\gamma}, \quad \gamma > 0$

For $\beta = 1$ consider the following cubic term belonging to $\tilde{\mathcal{L}}_N$:

$$C = \frac{1}{\sqrt{N}} \int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y))\varphi_t(y) a_x^* a_y^* a_x$$

Then

$$\langle \tilde{\mathcal{U}}_N\Omega, C\mathcal{U}_2\Omega\rangle = \frac{1}{\sqrt{N}} \int \mathrm{d}x\mathrm{d}y N^3 V(N(x-y))\varphi_t(y)\langle \underbrace{a_x a_y \tilde{\mathcal{U}}_N\Omega}_{\Psi_N(x,y)}, \underbrace{a_x\mathcal{U}_2\Omega}_{h(x)}\rangle$$

If $\Psi_N(x,y) = \sqrt{N}\, g(N(x-y))$ we have by scaling

$$\langle \tilde{\mathcal{U}}_N\Omega, C\mathcal{U}_2\Omega\rangle = \int \mathrm{d}x\mathrm{d}y V(x-y) g(x-y) h(x) \varphi_t(x - {}^{y}/_{N}) \simeq O(1)$$

This singular state is compatible with the condition $\langle \tilde{\mathcal{U}}_N\Omega, \mathcal{H}_N\tilde{\mathcal{U}}_N\Omega\rangle \sim O(1)$

Extension to the Gross-Pitaevskii regime & perspectives

In the Gross-Pitaevskii scaling, for initial data of the form $W(\sqrt{N}\varphi)T(k_0)\Omega$, fluctuations cannot be described by a quadratic generator

$$\|\left(e^{-it\mathcal{H}_N^{(GP)}} W(\sqrt{N}\varphi)T(k_0)\Omega - W(\sqrt{N}\varphi_t)\, T(k_t)\mathcal{U}_2\Omega\right)\| \nrightarrow 0$$

- Understand the role of cubic correlations, starting from the statics.

  Cubic correlations seem to play a role in proving the **Lee-Huang-Yang second order correction** to the ground state energy of the bose gas:

$$\lim_{\substack{N\to\infty \\ \rho=\text{const.}}} \frac{E_N}{N} = 4\pi\rho a\left[1 + \frac{128}{15\pi}(\rho a^3)^{1/2} + \ldots\right],$$

  see Erdös-Schlein-Yau (2008) and Yau-Yin (2013)

The limiting generator

The limit dynamics $\mathcal{U}_2(t)$ has quadratic generator:

$$\begin{aligned}\mathcal{L}_2(t) &= (i\partial_t T_t^*)T_t + \tfrac{1}{2}\int \mathrm{d}x\, \nabla_x a_x^* \nabla_x a_x \\ &+ \tfrac{1}{2}\int \mathrm{d}x\, \Big(2a_x^* a(-\Delta_x p_x) + a^*(\nabla_x p_x) a(\nabla_x p_x) + a^*(s_x) a(-\Delta_x s_x)\Big) \\ &+ \int \mathrm{d}x \Big(a^*(c_x) a^*(-\Delta_x r_x) + a^*(p_x) a^*(-\Delta_x k_x)\Big) \\ &+ \tfrac{1}{4}\int \mathrm{d}x\, \mathrm{d}y\, (1 - f_\ell(x-y)) \Big(\Delta_{\frac{x+y}{2}} \varphi_t^2((x+y)/2)\Big)\, a_x^* a_y^* \\ &+ \tfrac{3a_0}{\ell^3}\int \mathrm{d}x\, \mathrm{d}y\, \chi(|x-y| \le \ell)\, \varphi_t^2((x+y)/2)\, a_x^* a_y^* \\ &+ 4\pi a_0 \int \mathrm{d}x\, \varphi_t^2(x)\Big(a^*(c_x)a^*(p_x) + a_x^* a^*(p_x) + 2a^*(c_x)a(s_x) + a(s_x)a(s_x)\Big) \\ &+ b_0 \int \mathrm{d}x\, |\varphi_t(x)|^2 \Big(a^*(c_x)a(c_x) + 2a^*(c_x)a^*(s_x) + a^*(s_x)a(s_x)\Big)\end{aligned}$$

with $c(k_t) := \sum_{j=0}^{\infty} \frac{(k_t \bar{k}_t)^n}{2n!} = 1 + p(k_t)$; $s(k_t) := k_t + \sum_{j=1}^{\infty} \frac{(k_t \bar{k}_t)^n k_t}{(2n+1)!} = k_t + r(k_t)$.

Limit dynamics and Bogoliubov transformation

For every $t \in \mathbb{R}$ there exist $u_t, v_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ such that

$$\mathcal{U}_2^*(t)\, a(f)\, \mathcal{U}_2(t) = a(u_t f) + a^*(v_t f) := b(f)$$
$$\mathcal{U}_2^*(t)\, a^*(f)\, \mathcal{U}_2(t) = a(v_t f) + a^*(u_t f) := b^*(f)$$

with $b(f)$ and $b^*(g)$ satisfying canonical commutation relations.

The limit dynamics has three degrees of freedom: (φ_t, u_t, v_t)

If we define $\Theta_t = \begin{pmatrix} u_t & v_t \\ v_t & u_t \end{pmatrix}$ than $i\partial_t \Theta_t = D(t)\Theta_t$ with

$$D(t) = \begin{pmatrix} -\Delta + A_1 & A_2 \\ A_2 & -\Delta + A_1 \end{pmatrix}$$

and A_1 and A_2 operators on $L^2(\mathbb{R}^3)$, whose kernel can be explicitely written.
$\rightarrow$ (u_t, v_t) are determined by a partial differential equation on $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$

Correlation structure

$$\left\| \left(e^{-it\mathcal{H}_N^{(\beta)}} W(\sqrt{N}\varphi) T(k_0)\Omega - W(\sqrt{N}\varphi_t)\, T(k_t)\,\mathcal{U}_2\Omega \right) \right\| \leq C\, N^{-\gamma} e^{c_1 \exp(c_2|t|)}$$

Remark. The upper bound for the ground state energy of a Bose gas [Dyson 1957, Lieb-Seiringer-Yngvason 2000]

$$e_0(\rho) < 4\pi\rho\, a_0(1 + O(\rho a_0^3))$$

can be obtained using a trial function of the form

$$\Psi_N(x_1, \ldots, x_N) = \prod_{i<j} f_\ell(|x_i - x_j|)$$

where $f_\ell(x)$ is a solution of the scattering equation

$$(-\Delta + \tfrac{1}{2}V)f_\ell(x) = 0 \qquad \forall\ |x| \leq \ell$$

with boundary condition $f_\ell(\ell) = 1$. For the proof to be valid we need

$$\tfrac{4}{3}\pi\ell^3\rho = 1$$

Correlations affect the GP dynamics

Evolution of $\gamma^{(1)}_{N,t}$:

$$i\partial_t\gamma^{(1)}_{N,t} = [-\Delta, \gamma^{(1)}_{N,t}] + (N-1)\,\mathrm{Tr}_2[N^2 V(N(x_1-x_2)), \gamma^{(2)}_{N,t}]$$

In terms of the operator kernels:

$$i\partial_t\gamma^{(1)}_{N,t}(x_1;x_1') = (-\Delta_{x_1} + \Delta_{x_1'})\,\gamma^{(1)}_{N,t}(x_1;x_1')$$
$$+(N-1)\int \mathrm{d}x_2 N^2(V(N(x_1-x_2)) - V(N(x_1'-x_2)))\gamma^{(2)}_{N,t}(x_1,x_2;x_1',x_2)$$

The ansatz

$$\gamma^{(1)}_{N,t}(x_1;x_2) = \varphi_t(x_1)\overline{\varphi_t(x_2)}$$

$$\gamma^{(2)}_{N,t}(x_1,x_2;x_1',x_2') = \varphi_t(x_1)\varphi_t(x_2)\overline{\varphi_t(x_1')}\,\overline{\varphi_t(x_2')}$$

leads to

$$i\partial_t\varphi_t(x_1) = -\Delta\varphi_t(x_1) + \int \mathrm{d}x_2(N-1)N^2 V(N(x_1-x_2))|\varphi_t(x_2)|^2\varphi_t(x_1)$$

$$\xrightarrow[N\to\infty]{} -\Delta\varphi_t(x_1) + b_0\,|\varphi_t(x_1)|^2\varphi_t(x_1)$$

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The ansatz

$$\gamma^{(1)}_{N,t}(x_1; x_2) = \varphi_t(x_1)\overline{\varphi_t(x_2)}$$

$$\gamma^{(2)}_{N,t}(x_1, x_2; x_1', x_2') = f(N(x_1 - x_2))f(N(x_1' - x_2'))\varphi_t(x_1)\varphi_t(x_2)\overline{\varphi_t(x_1')}\,\overline{\varphi_t(x_2')}$$

leads to

$$i\partial_t \varphi_t(x_1) = -\Delta\varphi_t(x_1) + \int \mathrm{d}x_2 (N-1)N^2 V(N(x_1 - x_2))f(N(x_1 - x_2))|\varphi_t(x_2)|^2 \varphi_t(x_1)$$

$$\xrightarrow[N\to\infty]{} -\Delta\varphi_t(x_1) + 8\pi a_0\,|\varphi_t(x_1)|^2\varphi_t(x_1)$$